

ESSENTIAL SELF-ADJOINTNESS OF MAGNETIC SCHRÖDINGER OPERATORS ON LOCALLY FINITE GRAPHS

OGNJEN MILATOVIC

ABSTRACT. We give sufficient conditions for essential self-adjointness of magnetic Schrödinger operators on locally finite graphs. Two of the main theorems of the present paper generalize recent results of Torki-Hamza.

1. INTRODUCTION AND THE MAIN RESULTS

1.1. The setting. Let $G = (V, E)$ be an infinite graph without loops and multiple edges between vertices. By $V = V(G)$ and $E = E(G)$ we denote the set of vertices and the set of unoriented edges of G respectively. In what follows, the notation $m(x)$ indicates the degree of a vertex x , that is, the number of edges that meet at x . We assume that G is locally finite, that is, $m(x)$ is finite for all $x \in V$.

In what follows, $x \sim y$ indicates that there is an edge that connects x and y . We will also need a set of oriented edges

$$E_0 := \{[x, y], [y, x] : x, y \in V \text{ and } x \sim y\}. \quad (1.1)$$

The notation $e = [x, y]$ indicates an oriented edge e with starting vertex $o(e) = x$ and terminal vertex $t(e) = y$. The definition (1.1) means that every unoriented edge in E is represented by two oriented edges in E_0 . Thus, there is a two-to-one map $p: E_0 \rightarrow E$. For $e = [x, y] \in E_0$, we denote the corresponding reverse edge by $\widehat{e} = [y, x]$. This gives rise to an involution $e \mapsto \widehat{e}$ on E_0 .

To help us write formulas in unambiguous way, we fix an orientation on each edge by specifying a subset E_s of E_0 such that $E_0 = E_s \cup \widehat{E_s}$ (disjoint union), where $\widehat{E_s}$ denotes the image of E_s under the involution $e \mapsto \widehat{e}$. Thus, we may identify E_s with E by the map p .

In the sequel, we assume that G is connected, that is, for any $x, y \in V$ there exists a path γ joining x and y . Here, γ is a sequence $x_1, x_2, \dots, x_n \in V$ such that $x = x_1$, $y = x_n$, and $x_j \sim x_{j+1}$ for all $1 \leq j \leq n-1$. The length of a path γ is defined as the number of edges in γ .

The distance $d(x, y)$ between vertices x and y of G is defined as the number of edges in the shortest path connecting the vertices x and y . Fix a vertex $x_0 \in V$ and define $r(x) := d(x_0, x)$. The n -neighborhood $B_n(x_0)$ of $x_0 \in V$ is defined as

$$\{x \in V : r(x) \leq n\} \cup \{e = [x, y] \in E_s : r(x) \leq n \text{ and } r(y) \leq n\}. \quad (1.2)$$

In what follows, $C(V)$ is the set of complex-valued functions on V , and $C(E_s)$ is the set of functions $Y: E_0 \rightarrow \mathbb{C}$ such that $Y(e) = -Y(\widehat{e})$. The notations $C_c(V)$ and $C_c(E_s)$ denote the sets of finitely supported elements of $C(V)$ and $C(E_s)$ respectively.

In the sequel, we assume that V is equipped with a weight $w: V \rightarrow \mathbb{R}^+$. By $\ell_w^2(V)$ we denote the space of functions $f \in C(V)$ such that $\|f\| < \infty$, where $\|f\|$ is the norm corresponding to the inner product

$$(f, g) := \sum_{x \in V} w(x) f(x) \overline{g(x)}. \quad (1.3)$$

Additionally, we assume that E is equipped with a weight $a: E_0 \rightarrow \mathbb{R}^+$ such that $a(e) = a(\widehat{e})$ for all $e \in E_0$. This makes $G = (G, w, a)$ a weighted graph weights w and a .

1.2. Magnetic Schrödinger Operator. Let $U(1) := \{z \in \mathbb{C}: |z| = 1\}$ and $\sigma: E_0 \rightarrow U(1)$ with $\sigma(\widehat{e}) = \overline{\sigma(e)}$ for all $e \in E_0$, where \bar{z} denotes the complex conjugate of $z \in \mathbb{C}$.

We define the magnetic Laplacian $\Delta_\sigma: C(V) \rightarrow C(V)$ on the graph (G, w, a) by the formula

$$(\Delta_\sigma u)(x) = \frac{1}{w(x)} \sum_{e \in \mathcal{O}_x} a(e)(u(x) - \sigma(\widehat{e})u(t(e))), \quad (1.4)$$

where $x \in V$ and

$$\mathcal{O}_x := \{e \in E_0: o(e) = x\}. \quad (1.5)$$

For the case $a \equiv 1$ and $w \equiv 1$, the definition (1.4) is the same as in Dodziuk–Mathai [10]. For the case $\sigma \equiv 1$, see Sy–Sunada [29] and Torki–Hamza [30].

Let $q: V \rightarrow \mathbb{R}$, and consider a Schrödinger-type expression

$$Hu := \Delta_\sigma u + qu \quad (1.6)$$

We give sufficient conditions for $H|_{C_c(V)}$ to be essentially self-adjoint in the space $\ell_w^2(V)$. We first state the main results, and in Section 2 we make a few remarks concerning the existing work on the essential self-adjointness problem on locally finite graphs.

Theorem 1.3. *Assume that (G, w, a) is an infinite, locally finite, connected, oriented, weighted graph with $w(x) \equiv w_0$, where $w_0 > 0$ is a constant. Additionally, assume that there exists a constant $C \in \mathbb{R}$ such that $q(x) \geq -C$ for all $x \in V$. Then, the operator $H|_{C_c(V)}$ is essentially self-adjoint in $\ell_w^2(V)$.*

In the next theorem, we will need the following additional assumption on the graph G .

Assumption (A) Assume that

$$\lim_{n \rightarrow \infty} \frac{m_n a_n}{n^2} = 0, \quad (1.7)$$

where

$$m_n := \max_{x \in B_n(x_0)} (m(x)) \quad \text{and} \quad a_n := \max_{x \in B_n(x_0)} \left(\max_{e \sim x} \left(\frac{a(e)}{w(x)} \right) \right), \quad (1.8)$$

where $B_n(x_0)$ as in (1.2), and $e \sim x$, with $e \in E_s$ and $x \in V$, indicates that $t(e) = x$ or $o(e) = x$.

Theorem 1.4. *Assume that (G, w, a) is an infinite, locally finite, connected, oriented, weighted graph. Assume that the Assumption (A) is satisfied. Additionally, assume that there exists a constant $C \in \mathbb{R}$ such that*

$$(Hu, u) \geq -C\|u\|^2 \quad \text{for all } u \in C_c(V), \quad (1.9)$$

where (\cdot, \cdot) and $\|\cdot\|$ are as in (1.3). Then, the operator $H|_{C_c(V)}$ is essentially self-adjoint in $\ell_w^2(V)$.

In the next theorem, we will need the notion of weighted distance on G . Let $a: E_0 \rightarrow \mathbb{R}^+$ be as in (1.4). Following Colin de Verdière, Torki-Hamza, and Truc [5], we define the weighted distance $d_{w,a}$ on G as follows:

$$d_{w,a}(x, y) := \inf_{\gamma \in P_{x,y}} L_{w,a}(\gamma), \quad (1.10)$$

where $P_{x,y}$ is the set of all paths $\gamma: x = x_1, x_2, \dots, x_n = y$ such that $x_j \sim x_{j+1}$ for all $1 \leq j \leq n-1$, and the length $L_{w,a}(\gamma)$ is computed as follows:

$$L_{w,a}(\gamma) = \sum_{j=1}^{n-1} \sqrt{\frac{\min\{w(x_j), w(x_{j+1})\}}{a([x_j, x_{j+1}])}}. \quad (1.11)$$

We say that the metric space $(G, d_{w,a})$ is complete if every Cauchy sequence of vertices has a limit in V .

In what follows, we say that G is a graph of bounded degree if there exists a constant $N > 0$ such that $m(x) \leq N$ for all $x \in V$.

Theorem 1.5. *Assume that (G, w, a) is an infinite, locally finite, connected, oriented, weighted graph. Assume that G is a graph of bounded degree. Assume that $(G, d_{w,a})$ is a complete metric space. Additionally, assume that H satisfies (1.9). Then, the operator $H|_{C_c(V)}$ is essentially self-adjoint in $\ell_w^2(V)$.*

Remark 1.6. Let $d_{w,a}$ be as in (1.10). It is easily seen that if G is a graph of bounded degree and if (1.7) is satisfied, then $(G, d_{w,a})$ is complete.

The following example describes a graph G of bounded degree such that $(G, d_{w,a})$ is complete and (1.7) is not satisfied.

Examples . (i) Denote $\mathbb{Z}_+ := \{1, 2, 3, \dots\}$, and consider the graph $G_1 = (V, E)$ with $V = \mathbb{Z}_+ \cup \{0\}$ and $E = \{[n-1, n]: n \in \mathbb{Z}_+\}$. Define $a([n-1, n]) = n$ and $w(n-1) = \frac{1}{n}$, for all $n \in \mathbb{Z}_+$.

Since $w(x)$ is not constant, we cannot use Theorem 1.3 in this example.

Let $K \in \mathbb{Z}_+$ and let m_K and a_K be as in (1.8) with $n = K$ and $x_0 = 0$. We have $m_K = 2$ and $a_K = (K+1)^2$. Thus,

$$\lim_{K \rightarrow \infty} \frac{m_K a_K}{K^2} = 2,$$

and (1.7) is not satisfied. Thus, in this example, we cannot use Theorem 1.4.

Fix $K_0 \in \mathbb{Z}_+ \cup \{0\}$, and let $K > K_0$. For $x_0 = K_0$ and $x = K$, by (1.10) we have

$$d_{w,a}(x_0, x) = \sum_{n=K_0}^{K-1} \frac{1}{\sqrt{(n+1)(n+2)}} \rightarrow \infty, \quad \text{as } K \rightarrow \infty.$$

Thus, the metric $d_{w,a}$ is complete. Additionally, the graph G_1 has bounded degree. By Theorem 1.5 the operator $\Delta_\sigma|_{C_c(V)}$ is essentially self-adjoint in $\ell_w^2(V)$.

The following example describes a graph of unbounded degree such that (1.7) is satisfied.

(ii) Consider $G_2 = (V, E)$, where $V = \{x_0, x_1, x_2, \dots\}$. The vertices are arranged in a “triangular” pattern so that x_0 is in the first row, x_1 and x_2 are in the second row, x_3, x_4 , and x_5 are in the third row, and so on. The vertex x_0 is connected to x_1 and x_2 . The vertex x_i , where $i = 1, 2$, is connected to every vertex x_j , where $j = 3, 4, 5$. The pattern continues so that each of k vertices in the k -th row is connected to each of $k+1$ vertices in the $(k+1)$ -th row. Define $a(e) \equiv 1$ for all $e \in E$. For every vertex x in the n -th row, define $w(x) = n^{-1/2}$.

Since $w(x)$ is not constant, we cannot use Theorem 1.3. Since G_2 does not have a bounded degree, we cannot use Theorem 1.5.

Let $K \in \{1, 2, \dots\}$. Let m_K and a_K be as in (1.8) with $n = K$ and x_0 as in this example. We have $m_K = 2K + 2$ and $a_K = \sqrt{K+1}$. Thus,

$$\lim_{K \rightarrow \infty} \frac{m_K a_K}{K^2} = 0,$$

and (1.7) is satisfied. By Theorem 1.4 the operator $\Delta_\sigma|_{C_c(V)}$ is essentially self-adjoint in $\ell_w^2(V)$.

Remark 1.7. In the context of a not necessarily complete graph of bounded degree, a sufficient condition for essential-self adjointness of $\Delta_\sigma|_{C_c(V)}$ in $\ell_w^2(V)$ is given by Colin de Verdière, Torki-Hamza, and Truc [6, Theorem 3.1]. In the case $q \equiv 0$, Theorem 1.5 is contained in [6, Theorem 3.1].

2. BACKGROUND OF THE PROBLEM

In the context of a locally finite graph $G = (V, E)$, recently there has been a lot of interest in the operator

$$(\Delta u)(x) = \frac{1}{w(x)} \sum_{e \in \mathcal{O}_x} a(e)(u(x) - u(t(e))), \quad (2.1)$$

where $x \in V$ and \mathcal{O}_x is as in (1.5).

In many spectral-theoretic investigations of Δ and $\Delta + q$, where $q: V \rightarrow \mathbb{R}$ is a real-valued function, it is helpful to have a self-adjoint operator. Thus, finding sufficient conditions for essential self-adjointness of Δ and $\Delta + q$ is an important problem in analysis on locally finite graphs. Note that Δ in (2.1), also known as physical Laplacian, is generally an unbounded operator in $\ell_w^2(V)$. Putting $w \equiv 1$ and $a \equiv 1$ in (2.1) and dividing by the degree function $m(x)$, we get the normalized Laplacian, which is a bounded operator on $\ell_w^2(V)$, with inner product as in (1.3) with $w(x) = m(x)$. The normalized Laplacian has been studied extensively; see, for instance, Chung [4] and Mohar–Woess [21].

In the discussion that follows, the local finiteness assumption is understood, unless specified otherwise. The essential self-adjointness of $\Delta|_{C_c(V)}$, where Δ is as in (2.1) with $w \equiv 1$ and $a \equiv 1$, was proven by Wojciechowski [33] and Weber [31]. For Δ is as in (2.1) with $w \equiv 1$, the essential self-adjointness of $\Delta|_{C_c(V)}$ was proven by Jorgensen [14] (see also Jorgensen–Pearse [15]). With regard to Theorem 1.3 of the present paper, Torki-Hamza [30] proved the essential self-adjointness of $(\Delta + q)|_{C_c(V)}$, where Δ is as in (2.1) with $w \equiv c_0$ and $q \geq -c_1$, where $c_0 > 0$ and $c_1 \in \mathbb{R}$ are constants. The results of Wojciechowski [33], Weber [31], and Jorgensen [14] on the essential self-adjointness of Δ and the result of Torki-Hamza [30] on the essential self-adjointness of $(\Delta + q)|_{C_c(V)}$ with $q \geq -c_1$, where c_1 is a constant, are all contained in Keller–Lenz [17] and Keller–Lenz [18].

Under the assumption (1.7) above, the essential self-adjointness of $(d\delta + \delta d)|_{\Omega_0(G)}$, where $\Omega_0(G)$ denotes finitely supported forms $\alpha \in C(V) \oplus C(E)$, was proven by Masamune [19]. Additionally, Masamune [19] studied L^p -Liouville property for non-negative subharmonic forms on G .

In the context of a graph of bounded degree, Torki-Hamza [30] made an important link between the essential self-adjointness of $(\Delta + q)|_{C_c(V)}$, where Δ is as in (2.1) with $w \equiv 1$, and completeness of the weighted metric $d_{1,a}$ in (1.10) above; namely, if $d_{1,a}$ is complete and if $(\Delta + q)|_{C_c(V)}$ is semi-bounded below, then $(\Delta + q)|_{C_c(V)}$ is essentially self-adjoint on the space $\ell_w^2(V)$ with $w \equiv 1$. Theorem 1.5 of the present paper extends this result to the operator (1.6).

For a study of essential self-adjointness of $(\Delta + q)|_{C_c(V)}$ on a metrically non-complete graph, see Colin de Verdière, Torki-Hamza, and Truc [5]. Adjacency matrix operator on a locally finite graph was studied in Golénia [12]. For a study of the problem of deficiency indices for Schrödinger operators on a locally finite graph, see Golénia–Schumacher [13].

Kato’s inequality for Δ_σ as in (1.4), with $w \equiv 1$ and $a \equiv 1$, was proven in Dodziuk–Mathai [10] and used to study asymptotic properties of the spectrum of a certain discrete magnetic Schrödinger operator. For a study of essential self-adjointness of the magnetic Laplacian on a metrically non-complete graph, see Colin de Verdière, Torki-Hamza, and Truc [6]. A different model for discrete magnetic Laplacian was given by Sushch [28]. In the model of [28], the essential self-adjointness of a semi-bounded below discrete magnetic Schrödinger operator was proven.

Dodziuk [8], Wojciechowski [33], Wojciechowski [34], and Weber [31] explored connections between stochastic completeness and the essential self-adjointness of Δ . For extensions to the more general context of Dirichlet forms on discrete sets, see Keller–Lenz [17] and Keller–Lenz [18]. For a related study of random walks on infinite graphs, see Dodziuk [7], Dodziuk–Karp [9], Woess [32], and references therein.

Finally, we remark that the problem of essential self-adjointness of Schrödinger operators on infinite graphs has a strong connection to the corresponding problem on non-compact Riemannian manifolds; see Gaffney [11], Oleinik [22], Oleinik [23], Braverman [1], Shubin [25], Shubin [26], and [2].

3. PRELIMINARIES

In what follows, $d: C(V) \rightarrow C(E_s)$ is the standard differential

$$du(e) := u(t(e)) - u(o(e)).$$

The deformed differential $d_\sigma: C(V) \rightarrow C(E_s)$ is defined as

$$(d_\sigma u)(e) := \overline{\sigma(e)}u(t(e)) - u(o(e)), \quad \text{for all } u \in C(V), \quad (3.1)$$

where σ is as in (1.4).

The deformed co-differential $\delta_\sigma: C(E_s) \rightarrow C(V)$ is defined as follows:

$$(\delta_\sigma Y)(x) := \frac{1}{w(x)} \sum_{\substack{e \in E_s \\ t(e)=x}} \sigma(e)a(e)Y(e) - \frac{1}{w(x)} \sum_{\substack{e \in E_s \\ o(e)=x}} a(e)Y(e), \quad (3.2)$$

for all $Y \in C(E_s)$, where σ , w , and a are as in (1.4).

Let $\ell_a^2(E_s)$ denote the space of functions $F \in C(E_s)$ such that $\|F\| < \infty$, where $\|F\|$ is the norm corresponding to the inner product

$$(F, G) := \sum_{e \in E_s} a(e)F(e)\overline{G(e)}.$$

For a general background on the theory of magnetic Laplacian on graphs, see Mathai–Yates [20] and Sunada [27].

Lemma 3.1. *The following equality holds:*

$$(d_\sigma u, Y) = (u, \delta_\sigma Y), \quad \text{for all } u \in \ell_w^2(V), Y \in C_c(E_s), \quad (3.3)$$

where (\cdot, \cdot) on the left-hand side (right-hand side) denotes the inner product in $\ell_a^2(E_s)$ (in $\ell_w^2(V)$).

Proof. Using (3.1) and (3.2) we have

$$\begin{aligned} (u, \delta_\sigma Y) &= \sum_{x \in V} u(x) \left(\sum_{\substack{e \in E_s \\ t(e)=x}} a(e)\overline{\sigma(e)}Y(e) - \sum_{\substack{e \in E_s \\ o(e)=x}} a(e)\overline{Y(e)} \right) \\ &= \sum_{e \in E_s} a(e)u(t(e))\overline{\sigma(e)}Y(e) - \sum_{e \in E_s} a(e)u(o(e))\overline{Y(e)} \\ &= \sum_{e \in E_s} a(e)(\overline{\sigma(e)}u(t(e)) - u(o(e)))\overline{Y(e)} = (d_\sigma u, Y). \end{aligned}$$

The convergence of the sums is justified by observing that only finitely many $x \in V$ contribute to the sum as Y has finite support. \square

Using the definitions (3.1) and (3.2) together with the properties $a(\hat{e}) = a(e)$, $\sigma(\hat{e}) = \overline{\sigma(e)}$, and $|\sigma(e)| = 1$, which hold for all $e \in E_0$, one can easily prove the following lemma.

Lemma 3.2. *The equality $\delta_\sigma d_\sigma u = \Delta_\sigma u$ holds for all $u \in C(V)$.*

The following lemma follows easily from Lemma 3.2 and (3.3).

Lemma 3.3. *The operator $\Delta_\sigma|_{C_c(V)}$ is symmetric in $\ell_w^2(V)$:*

$$(\Delta_\sigma u, v) = (u, \Delta_\sigma v), \quad \text{for all } u, v \in C_c(V).$$

Lemma 3.4. *For all $u, v \in C(V)$ the following property holds:*

$$\begin{aligned} (\Delta_\sigma(uv))(x) &= (\Delta_\sigma u)(x)v(x) \\ &\quad + \frac{1}{w(x)} \sum_{e \in \mathcal{O}_x} a(e)\sigma(\widehat{e})u(t(e))(v(x) - v(t(e))), \end{aligned} \quad (3.4)$$

where $x \in V$ and \mathcal{O}_x is as in (1.5).

Proof. Using the definition (1.4) we have

$$\begin{aligned} (\Delta_\sigma(uv))(x) &= \frac{1}{w(x)} \sum_{e \in \mathcal{O}_x} a(e)(u(x)v(x)) \\ &\quad - \frac{1}{w(x)} \sum_{e \in \mathcal{O}_x} a(e)\sigma(\widehat{e})u(t(e))v(t(e)). \end{aligned} \quad (3.5)$$

Adding and subtracting

$$\frac{1}{w(x)} \sum_{e \in \mathcal{O}_x} a(e)\sigma(\widehat{e})u(t(e))v(x)$$

on the right-hand side of (3.5) and grouping the terms appropriately, we get (3.4). \square

In the proof of the following proposition, we will use a technique similar to Shubin [26, Section 5.1], Masamune [19], and Torki-Hamza [30].

Proposition 3.5. *Assume that $u \in \ell_w^2(V)$ and $Hu = 0$. Then the following holds for all $\phi \in C_c(V)$:*

$$\begin{aligned} &(H(u\phi), u\phi) \\ &= \sum_{e \in E_s} a(e)\sigma_1(\widehat{e})[u_1(t(e))u_1(o(e)) + u_2(t(e))u_2(o(e))](\phi(o(e)) - \phi(t(e)))^2 \\ &\quad + \sum_{e \in E_s} a(e)\sigma_2(\widehat{e})[-u_1(o(e))u_2(t(e)) + \\ &\quad + u_1(t(e))u_2(o(e))](\phi(o(e)) - \phi(t(e)))^2, \end{aligned} \quad (3.6)$$

where $u_1 := \operatorname{Re} u$, $u_2 := \operatorname{Im} u$, $\sigma_1 := \operatorname{Re} \sigma$, and $\sigma_2 := \operatorname{Im} \sigma$.

Proof. Using (3.4) with $v = \phi$, we obtain

$$\begin{aligned} (H(u\phi))(x) &= (Hu)(x)\phi(x) \\ &\quad + \frac{1}{w(x)} \sum_{e \in \mathcal{O}_x} a(e)\sigma(\widehat{e})u(t(e))(\phi(x) - \phi(t(e))). \end{aligned} \quad (3.7)$$

Taking the inner product (\cdot, \cdot) with $u\phi$ on both sides of (3.7), we obtain:

$$\begin{aligned} (H(u\phi), u\phi) &= (\phi(Hu), u\phi) \\ &+ \sum_{x \in V} \sum_{e \in \mathcal{O}_x} a(e)\sigma(\widehat{e})u(t(e))(\phi(x) - \phi(t(e)))\overline{u(x)}\phi(x). \end{aligned} \quad (3.8)$$

Taking the real parts on both sides of (3.8), we get

$$\begin{aligned} (H(u\phi), u\phi) &= \operatorname{Re} (\phi(Hu), u\phi) \\ &+ \operatorname{Re} \left(\sum_{x \in V} \sum_{e \in \mathcal{O}_x} a(e)\sigma(\widehat{e})u(t(e))(\phi(x) - \phi(t(e)))\overline{u(x)}\phi(x) \right). \end{aligned} \quad (3.9)$$

Since $\sigma(\widehat{e}) = \overline{\sigma(e)}$, it follows that $\sigma_1(\widehat{e}) = \sigma_1(e)$ and $\sigma_2(\widehat{e}) = -\sigma_2(e)$. Substituting $u = u_1 + iu_2$, $\sigma = \sigma_1 + i\sigma_2$ and $Hu = 0$ in (3.9) leads to

$$(H(u\phi), u\phi) = J_1 + J_2, \quad (3.10)$$

where

$$\begin{aligned} J_1 &:= \sum_{x \in V} \sum_{e \in \mathcal{O}_x} a(e)\sigma_1(\widehat{e})[u_1(t(e))u_1(x) + \\ &+ u_2(t(e))u_2(x)](\phi^2(x) - \phi(x)\phi(t(e))), \end{aligned}$$

and

$$\begin{aligned} J_2 &:= \sum_{x \in V} \sum_{e \in \mathcal{O}_x} a(e)\sigma_2(\widehat{e})[-u_1(x)u_2(t(e)) + \\ &+ u_1(t(e))u_2(x)](\phi^2(x) - \phi(x)\phi(t(e))). \end{aligned}$$

In each of the sums J_1 and J_2 an edge $e = [x, y] \in E_0$ occurs twice: once as $[x, y]$ and once as $[y, x]$. Since $a([x, y]) = a([y, x])$, $\sigma_1([x, y]) = \sigma_1([y, x])$, and $\sigma_2([x, y]) = -\sigma_2([y, x])$, it follows that the expressions

$$\begin{aligned} &a(e)\sigma_1(\widehat{e})(u_1(t(e))u_1(x) + u_2(t(e))u_2(x)) \\ &= a([x, y])\sigma_1([y, x])(u_1(y)u_1(x) + u_2(y)u_2(x)) \end{aligned}$$

and

$$\begin{aligned} &a(e)\sigma_2(\widehat{e})(-u_1(x)u_2(t(e)) + u_1(t(e))u_2(x)) \\ &= a([x, y])\sigma_2([y, x])(-u_1(x)u_2(y) + u_1(y)u_2(x)) \end{aligned}$$

are invariant under the involution $e \mapsto \widehat{e}$. Hence, in the sum J_1 , the contribution of $e = [x, y]$ and $\widehat{e} = [y, x]$ together is

$$a(e)\sigma_1(\widehat{e})(u_1(t(e))u_1(x) + u_2(t(e))u_2(x))(\phi(x) - \phi(t(e)))^2. \quad (3.11)$$

In the sum J_2 , the contribution of $e = [x, y]$ and $\widehat{e} = [y, x]$ together is

$$a(e)\sigma_2(\widehat{e})(-u_1(x)u_2(t(e)) + u_1(t(e))u_2(x))(\phi(x) - \phi(t(e)))^2. \quad (3.12)$$

Using (3.11) and (3.12), we can rewrite (3.10) to get (3.6). \square

We now give the definitions of minimal and maximal operators associated with the expression (1.6).

3.6. Operators H_{\min} and H_{\max} . We define the operator H_{\min} by the formula

$$H_{\min}u := Hu, \quad \text{Dom}(H_{\min}) := C_c(V). \quad (3.13)$$

Since q is real-valued, the following lemma follows easily from Lemma 3.3.

Lemma 3.7. *The operator H_{\min} is symmetric in $\ell_w^2(V)$.*

We define $H_{\max} := (H_{\min})^*$, where T^* denotes the adjoint of operator T . We also define $\mathcal{D} := \{u \in \ell_w^2(V) : Hu \in \ell_w^2(V)\}$.

Lemma 3.8. *The following hold: $\text{Dom}(H_{\max}) = \mathcal{D}$ and $H_{\max}u = Hu$ for all $u \in \mathcal{D}$.*

Proof. Suppose that $v \in \mathcal{D}$. Then, for all $u \in C_c(V)$ we have

$$(H_{\min}u, v) = (\Delta_\sigma u + qu, v) = (u, \Delta_\sigma v + qv).$$

Since $(\Delta_\sigma v + qv) \in \ell_w^2(V)$, by the definition of the adjoint we obtain $v \in \text{Dom}((H_{\min})^*)$ and $(H_{\min})^*v = \Delta_\sigma v + qv$. This shows that $\mathcal{D} \subset \text{Dom}((H_{\min})^*)$ and $(H_{\min})^*v = Hv$ for all $v \in \mathcal{D}$.

Suppose that $v \in \text{Dom}((H_{\min})^*)$. Then, there exists $z \in \ell_w^2(V)$ such that

$$(\Delta_\sigma u + qu, v) = (u, z), \quad \text{for all } u \in C_c(V). \quad (3.14)$$

Since $(\Delta_\sigma u + qu, v) = (u, \Delta_\sigma v + qv)$ and since $C_c(V)$ is dense in $\ell_w^2(V)$, from (3.14) it follows that $\Delta_\sigma v + qv = z = (H_{\min})^*v$. This shows that $\text{Dom}((H_{\min})^*) \subset \mathcal{D}$. Thus, we have shown that $\mathcal{D} = \text{Dom}((H_{\min})^*)$ and $(H_{\min})^*v = Hv$ for all $v \in \mathcal{D}$. \square

4. PROOF OF THEOREM 1.3

We begin with a version of Kato's inequality for discrete magnetic Laplacian. For the original version in the setting of differential operators, see Kato [16]. In the case $w \equiv 1$ and $a \equiv 1$, the following lemma was proven in Dodziuk–Mathai [10].

Lemma 4.1. *Let Δ and Δ_σ be as in (2.1) and (1.4) respectively. Then, the following pointwise inequality holds for all $u \in C(V)$:*

$$|u| \cdot \Delta|u| \leq \text{Re} (\Delta_\sigma u \cdot \bar{u}), \quad (4.1)$$

where $\text{Re } z$ denotes the real part of a complex number z .

Proof. Using (2.1), (1.4), and the property $|\sigma(\hat{e})| \leq 1$, we obtain

$$\begin{aligned} & (|u| \cdot \Delta|u|)(x) - \text{Re} (\Delta_\sigma u \cdot \bar{u})(x) \\ &= \frac{1}{w(x)} \sum_{e \in \mathcal{O}_x} a(e) \text{Re}(\sigma(\hat{e})u(t(e))\overline{u(x)} - |u(x)||u(t(e))|) \leq 0, \end{aligned}$$

and the lemma is proven. \square

Continuation of the Proof of Theorem 1.3. Without loss of generality, we may assume $w(x) \equiv w_0 = 1$. By adding a constant to q , we may assume that $q(x) \geq 1$, for all $x \in V$. Let H_{\min} and H_{\max} be as in Section 3.6.

Since $H_{\min} = H|_{C_c(V)}$ is symmetric and since $(H_{\min}u, u) \geq \|u\|^2$, for all $u \in C_c(V)$, the essential self-adjointness of H_{\min} is equivalent to the following statement: $\ker(H_{\max}) = \{0\}$; see Reed–Simon [24, Theorem X.26]. Let $u \in \text{Dom}(H_{\max})$ satisfy $H_{\max}u = 0$:

$$(\Delta_\sigma + q)u = 0. \quad (4.2)$$

By (4.1) and (4.2) we get the pointwise inequality

$$|u| \cdot \Delta|u| \leq \text{Re}(\Delta_\sigma u \cdot \bar{u}) = \text{Re}(-qu \cdot \bar{u}) = -q|u|^2 \leq -|u|^2. \quad (4.3)$$

Rewriting (4.3) we obtain the pointwise inequality

$$|u|(\Delta|u| + |u|) \leq 0,$$

which leads to

$$0 \geq (\Delta|u|)(x) + |u(x)| = \sum_{e \in \mathcal{O}_x} a(e)(|u(x)| - |u(t(e))|) + |u(x)|, \quad (4.4)$$

for all $x \in V$.

From here on, the argument is the same as in Torki-Hamza [30, Theorem 3.1]. Assume that there exists $x_0 \in V$ such that $|u(x_0)| > 0$. Then, by (4.4) with $x = x_0$, there exists $x_1 \in V$ such that $|u(x_0)| < |u(x_1)|$. Using (4.4) with $x = x_1$, we see that there exists $x_2 \in V$ such that $|u(x_2)| > |u(x_1)|$. Continuing like this, we get a strictly increasing sequence of positive real numbers $|u(x_n)|$. But this contradicts the fact that $|u| \in \ell_w^2(V)$. Hence, $|u| \leq 0$ for all $x \in V$. In other words, $u = 0$. \square

5. PROOF OF THEOREM 1.4

In what follows, we will use a sequence of cut-off functions.

5.1. Cut-off functions. Fix a vertex $x_0 \in V$, and define

$$\phi_n(x) := \left(\left(\frac{2n - r(x)}{n} \right) \vee 0 \right) \wedge 1, \quad x \in V, \quad n \in \mathbb{Z}_+, \quad (5.1)$$

where $r(x) = d(x_0, x)$ is as in Section 1.1.

As shown in Masamune [19, Proposition 3.2], the sequence $\{\phi_n\}_{n \in \mathbb{Z}_+}$ satisfies the following properties:

- (i) $0 \leq \phi_n(x) \leq 1$, for all $x \in V$;
- (ii) $\phi_n(x) = 1$ for $x \in B_n(x_0)$, and $\phi_n(x) = 0$ for $x \notin B_{2n}(x_0)$;
- (iii) $\sup_{e \in E_s} |(d\phi_n)(e)| \leq \frac{1}{n}$.

Continuation of the Proof of Theorem 1.4. We will use a technique similar to Shubin [26, Section 5.1], Masamune [19], and Torki-Hamza [30].

Since H satisfies (1.9), without loss of generality, we may add $(C+1)I$ to H and assume that

$$(Hv, v) \geq \|v\|^2, \quad \text{for all } v \in C_c(V). \quad (5.2)$$

Since $H_{\min} = H|_{C_c(V)}$ is symmetric and satisfies (5.2), the essential self-adjointness of H_{\min} is equivalent to the following statement: $\ker(H_{\max}) = \{0\}$; see Reed–Simon [24, Theorem X.26].

Let $u \in \text{Dom}(H_{\max})$ satisfy $H_{\max}u = 0$. Let ϕ_n be as in Section 5.1. Starting from (3.6) with $\phi = \phi_n$ and using the properties (ii) and (iii) of ϕ_n , together with $|\sigma_1| \leq 1$ and $|\sigma_2| \leq 1$, we get the following estimate:

$$\begin{aligned} & (H(u\phi_n), u\phi_n) \\ & \leq \frac{1}{n^2} \sum_{e \in B_{2n}(x_0)} a(e)(u_1^2(t(e)) + u_1^2(o(e)) + u_2^2(t(e)) + u_2^2(o(e))), \end{aligned} \quad (5.3)$$

where $B_{2n}(x_0)$ is as in property (ii) of ϕ_n .

By (1.8) and (5.3) we obtain

$$\begin{aligned} (H(u\phi_n), u\phi_n) & \leq \frac{m_{2n}a_{2n}}{n^2} \sum_{x \in B_{2n}(x_0)} w(x)((u_1(x))^2 + (u_2(x))^2) \\ & \leq \frac{m_{2n}a_{2n}}{n^2} \|u\|^2. \end{aligned} \quad (5.4)$$

Since $\phi_n u \in C_c(V)$, the inequality (5.2) is satisfied with $v = \phi_n u$. Combining (5.4) and (5.2) we get

$$\|u\phi_n\|^2 \leq \frac{m_{2n}a_{2n}}{n^2} \|u\|^2. \quad (5.5)$$

We now take the limit as $n \rightarrow \infty$ in (5.5). Using the assumption (1.7) and the definition of ϕ_n , we obtain $\|u\|^2 \leq 0$. This shows that $u = 0$. \square

6. PROOF OF THEOREM 1.5

In the case $w \equiv 1$, the following family of cut-off functions was constructed in Torki-Hamza [30].

6.1. Family of cut-off functions. Fix $x_0 \in V$. For $R > 0$ define

$$U_R := \{x \in V : d_{w,a}(x_0, x) \leq R\}, \quad (6.1)$$

where $d_{w,a}$ is as in (1.10). Define

$$\psi_R := \min\{1, d_{w,a}(x, V \setminus U_{R+1})\}. \quad (6.2)$$

The family ψ_R satisfies the following properties:

- (i) $\psi_R(x) \equiv 1$, for all $x \in U_R$; (ii) $\psi_R(x) \equiv 0$, for all $x \in V \setminus U_{R+1}$; (iii) $0 \leq \psi_R \leq 1$, for all $x \in V$;
- (iv) ψ_R has finite support;
- (v) ψ_R is a Lipschitz function with Lipschitz constant 1.

It is easy to see that the properties (i), (ii), (iii) and (v) hold. To prove property (iv), we will show that U_{R+1} is finite. Clearly, U_{R+1} is a closed and bounded set. With $d_{w,a}$ defined as in (1.10), it follows that $(V, d_{w,a})$ is a length space in the sense of Burago–Burago–Ivanov [3, Section 2.1]. Additionally, we know by hypothesis that $(V, d_{w,a})$ is complete. Thus, by [3, Theorem 2.5.28] the set U_{R+1} is compact. Suppose that there exists a sequence of vertices $\{x_n\}_{n \in \mathbb{Z}_+} \subset U_{R+1}$. Since U_{R+1} is compact, there exists a subsequence, which we again denote by $\{x_n\}_{n \in \mathbb{Z}_+}$, such that $x_n \rightarrow x$ and $x \in U_{R+1}$. Let $F = \{y_1, y_2, \dots, y_s\}$ be the set of all vertices $y \in V$ such that there is an edge connecting y and x . The set F is finite since G is locally finite. Let $k_0 = \max\{n: x_n \in F\}$ (if there is no x_n such that $x_n \in F$, we take $k_0 = 0$). Take $\epsilon > 0$ such that $\epsilon < \min_{1 \leq j \leq s} (d_{w,a}(y_j, x))$. Then there exists $n_0 \in \mathbb{Z}_+$ such that $d_{w,a}(x_n, x) < \epsilon$ for all $n \geq n_0$. Take $K \in \mathbb{Z}_+$ such that $K > \max\{k_0, n_0\}$. Clearly, $d_{w,a}(x_K, x) < \epsilon$. Since $(V, d_{w,a})$ is a complete locally compact length space, by [3, Theorem 2.5.23] there is a shortest path γ connecting x_K and x . This means that the length $L_{w,a}(\gamma)$ of the path γ satisfies

$$L_{w,a}(\gamma) = d_{w,a}(x_K, x) < \epsilon. \quad (6.3)$$

Since $x_K \notin F$, there is no edge connecting x_K and x . Hence, the path γ will contain a vertex $y_j \in F$. Thus, $L_{w,a}(\gamma) > d_{w,a}(y_j, x) > \epsilon$, and this contradicts (6.3). Hence, the set U_{R+1} is finite.

Continuation of the Proof of Theorem 1.5. We adapt the technique of Torki-Hamza [30] to our setting.

As in the proof of Theorem 1.4, without the loss of generality, we will assume (5.2) and show that $\ker(H_{\max}) = \{0\}$. Let $u \in \text{Dom}(H_{\max})$ satisfy $H_{\max}u = 0$. Using (3.6) with $\phi = \psi_R$, we get

$$\begin{aligned} & (H(u\psi_R), u\psi_R) \\ &= \frac{1}{2} \sum_{x \in V} \sum_{e \in \mathcal{O}_x} a(e) \sigma_1(\widehat{e}) [u_1(t(e))u_1(o(e)) + \\ & \quad + u_2(t(e))u_2(o(e))] (\psi_R(o(e)) - \psi_R(t(e)))^2 \\ & \quad + \frac{1}{2} \sum_{x \in V} \sum_{e \in \mathcal{O}_x} a(e) \sigma_2(\widehat{e}) [-u_1(o(e))u_2(t(e)) + \\ & \quad + u_1(t(e))u_2(o(e))] (\psi_R(o(e)) - \psi_R(t(e)))^2, \end{aligned} \quad (6.4)$$

where \mathcal{O}_x is as in (1.5). Using the inequality $2\alpha\beta \leq \alpha^2 + \beta^2$, properties $|\sigma_1| \leq 1$ and $|\sigma_2| \leq 1$, and the invariance of $a(e)$ and

$$(u_j^2(t(e)) + u_j^2(o(e))) (\psi_R(o(e)) - \psi_R(t(e)))^2, \quad j = 1, 2$$

under involution $e \mapsto \widehat{e}$, we get

$$\begin{aligned} & (H(u\psi_R), u\psi_R) \\ & \leq \frac{1}{2} \sum_{x \in V} \sum_{e \in \mathcal{O}_x} a(e) (u_1^2(o(e)) + u_2^2(o(e))) (\psi_R(o(e)) - \psi_R(t(e)))^2. \end{aligned} \quad (6.5)$$

Using properties (i), (ii) and (v) of ψ_R , (6.5) leads to

$$\begin{aligned} & (H(u\psi_R), u\psi_R) \\ & \leq \frac{1}{2} \sum_{x \in U_{R+1} \setminus U_R} \sum_{e \in \mathcal{O}_x} a(e) |u(o(e))|^2 (d_{w,a}(o(e), t(e)))^2. \end{aligned} \quad (6.6)$$

By (1.10) and (1.11) it follows that

$$d_{w,a}(o(e), t(e)) \leq \sqrt{\frac{w(o(e))}{a(e)}}. \quad (6.7)$$

Using (6.6), (6.7), and bounded degree assumption on G , we get

$$(H(u\psi_R), u\psi_R) \leq \frac{N}{2} \sum_{x \in U_{R+1} \setminus U_R} w(x) |u(x)|^2. \quad (6.8)$$

By property (iv) of ψ_R , it follows that $\psi_R u \in C_c(V)$; hence, the inequality (5.2) is satisfied with $v = \psi_R u$. Combining (6.8) and (5.2) we get

$$\|u\psi_R\|^2 \leq \frac{N}{2} \sum_{x \in U_{R+1} \setminus U_R} w(x) |u(x)|^2. \quad (6.9)$$

We now take the limit as $R \rightarrow \infty$ in (6.9). Using the definition of ψ_R and the assumption $u \in \ell_w^2(V)$, we obtain $\|u\|^2 \leq 0$. This shows that $u = 0$. \square

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DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF NORTH FLORIDA, JACKSONVILLE, FL 32224, USA

E-mail address: omilatov@unf.edu